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# 1 Coordinates, Symmetry, and Conservation Laws in Classical Mechanics

This document explores the relationship between coordinate changes, symmetry, and quantities that are conserved during the time evolution of classical systems. But before we can examine this relationship in detail, we must first develop a means for describing the dynamics of a general classical system. A Lagrangian formulation of the dynamics will prove to be a particularly natural way to elucidate these connections.

## 1.1 Hamilton's Principle

We begin by considering a system consisting of finitely many continuous degrees of freedom, and we assume that the state of our system can be described by $N$ coordinates $q_1, \ldots, q_n$ which we can denote in compact form by the vector $\mathbf{q}$.[1] The behavior of our system as it evolves through time can then be completely described by the function $\mathbf{q}(t) = (q_1(t), \ldots, q_N(t))$, which represents a path through this configuration space.

We now specify the dynamics of our system. We assume that the knowledge of $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t) = d\mathbf{q}/dt$ at any particular time $t$ completely characterizes the path of the system for all future and past times. Furthermore, we assume that there exists a function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ such that the action functional

$$S = \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, dt \tag{1}$$

is stationary along the actual path followed by the system.

This stationarity condition (known as Hamilton's principle) is an elegant and compact way to specify the dynamics of the system. However, in its current form it cannot easily be inverted to find $\mathbf{q}(t)$. The calculus of variations lets us transform this integral statement into a set of differential equations for $\mathbf{q}(t)$, which are much easier to solve (or at the very least, simulate).

The transformation proceeds as follows. We begin by calculating the action for an alternate path $\mathbf{q}'(t) = \mathbf{q}(t) = \delta\mathbf{q}(t)$ that differs from the true path by an infinitesimal perturbation $\delta\mathbf{q}(t)$ that vanishes at $t_1$ and $t_2$. (The perturbation in the velocity is simply given by $\dot{\mathbf{q}}'(t) = \dot{\mathbf{q}}(t) + \frac{d}{dt}\delta\mathbf{q}(t)$). Since the action is stationary for the true path, the first order variation of $S$ must vanish:

$$\delta S = \int_{t_1}^{t_2} L(\mathbf{q}'(t), \dot{\mathbf{q}}'(t), t)\, dt - \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, dt = 0 \tag{2}$$

[1] This notation is simply for convenience, and does not imply that $\mathbf{q}$ transforms as a physical vector.

We can then calculate $\delta S$ in terms of $\delta \mathbf{q}$:

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} L\left(\mathbf{q}+\delta\mathbf{q}, \dot{\mathbf{q}}+\frac{d}{dt}\delta\mathbf{q}, t\right) - L(\mathbf{q},\dot{\mathbf{q}},t)\,dt \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i\,dt \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)dt + \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i\,dt \\
&= \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i\,dt
\end{aligned}$$

where the first term in the second to last line vanishes because the perturbations vanish at the endpoints. Since the perturbation is effectively arbitrary, $\delta S$ vanishes if and only if

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0\,, \quad i = 1,\ldots,N \tag{3}$$

### Continuous Systems

Continuous systems generalize the finite systems considered above to the case where an inifnite number of degrees of freedom can be indexed by points $\mathbf{x}$ in a continuous space. In this case, we would write $q_i(t) = q(\mathbf{x}, t)$, but according to tradition we typically use the symbol $\phi(\mathbf{x}, t)$ and refer to this collection of degrees of freedom as a field. In addition to time derivatives $\dot{\phi}(\mathbf{x}, t) = \partial_t\phi$, we now have spatial derivatives $\partial_i\phi$ for $i = 1,\ldots,d$ where $d$ is the dimension of the vector $\mathbf{x}$.

We assume a certain degree of additivity in the lagriangian for this system, so that it can be written as an integral of a lagrangian density $\mathcal{L}$ that is a function only of the fields, their first derivatives (spatial and time), and spacetime:

$$L = \int \mathcal{L}(\phi, \partial_\alpha\phi, x^\alpha)\,d^d x$$

The action is then given by

$$S = \int_{t_1}^{t_2} L\,dt = \int \mathcal{L}(\phi, \partial_\alpha\phi, x^\alpha)\,d^{d+1}x$$

A derivation similar to the one given above for finite systems yields the differential equations:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} = 0 \tag{4}$$

**Mixed discrete-continuous systems**

Obviously, we can have a discrete number of continuous fields $\vec{\phi}(\mathbf{x}, t)$ as well, and the Euler-Lagrange equations become

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i)} = 0\,, \quad i = 1, \ldots, N \tag{5}$$

## 1.2 Coordinate Transformations

In the previous section, we assumed that a Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ exists for each set of coordinate set $(\mathbf{q}, \dot{\mathbf{q}}, t)$, but we did not say how we would construct $L$ in general. This raises the natural question: if we know the form of the Lagrangian in one frame of reference, how do we construct the Lagrangian in other frames of reference?

We assume that we know the Lagriangian for the generalized coordinates $\mathbf{q}$, and we seek the Lagrangian for another set of generalized coordinates $\mathbf{q}'$ related by the bijection

$$t' = F(t)\,, \quad q' = G(q, t) \tag{6}$$

whose inverses are denoted by

$$t = f(t')\,, \quad q = g(q', t')$$

With these definitions, we find that

$$q(t) = g(q'(F(t)), F(t))\,, \quad \dot{q}_i(t) = \frac{\partial g}{\partial q'_i} \dot{q'}_i \frac{\partial F}{\partial t} + \frac{\partial g}{\partial t'} \frac{\partial F}{\partial t}$$

but by the inverse function theorem we also have $\partial F / \partial t = (\partial f / \partial t')^{-1}$. Then by a straightforward application of the change of variables theorem, the action can be written as an integral over $t'$:

$$S = \int_{F([t_1, t_2])} L\left(g(q', t'), \frac{\partial g}{\partial q'} \left(\frac{\partial f}{\partial t'}\right)^{-1} \dot{q'} + \frac{\partial g}{\partial t'} \left(\frac{\partial f}{\partial t'}\right)^{-1}, f(t')\right) \left|\frac{\partial f}{\partial t'}\right| dt' \tag{7}$$

Thus, we see that the Lagrangian in this new coordinate system is given by

$$L'(q'(t'), \dot{q'}(t'), t') = L\left(g(q', t'), \frac{\partial g}{\partial q'} \left(\frac{\partial f}{\partial t'}\right)^{-1} \dot{q'} + \frac{\partial g}{\partial t'} \left(\frac{\partial f}{\partial t'}\right)^{-1}, f(t')\right) \left|\frac{\partial f}{\partial t'}\right| \tag{8}$$

**Note:** The coordinate transformations we consider here are often referred to as *passive transformations* in the physics literature because the system itself is unchanged during the transformation — only the internal coordinate system of the observer is modified.

**Continuous Systems**

This treatment can easily be extended to continuous systems, with

$$x' = F(x) \qquad \phi' = G(\phi, x)$$
$$x = f(x') \qquad \phi = g(\phi', x')$$

and

$$\phi(x) = g(\phi'(F(x)), F(x)), \quad \partial_\alpha \phi_i = (\partial_j g_i)(\partial_\beta \phi_j)(\partial_\alpha F_\beta) + (\partial_\beta g_i)(\partial_\alpha F_\beta)$$

Again, by applying the change of variables theorem, we see that

$$S = \int_{F(V)} d^d x \mathcal{L}'(\phi', \partial_\alpha \phi', x') \tag{9}$$

with

$$\mathcal{L}'(\phi'(x'), \partial_\alpha \phi'(x'), x') = \mathcal{L}\left(g(\phi'(x'), x'), (\partial_j g_i)(\partial_\beta \phi_j)(\partial_\alpha F_\beta) + (\partial_\beta g_i)(\partial_\alpha F_\beta), f(x')\right) |\mathrm{det} Df| \tag{10}$$

## 1.3 Symmetries and Conservation Laws

**Symmetry**

We say that a coordinate transformation represents a symmetry of the system if no experiment can be done to differentiate between the two coordinate systems. In other words, the laws of physics (i.e., the equations of motion for the system) are the same. This occurs when the transformed Lagrangian is equal to the original Lagrangian up to an additive factor of a total time derivative of a scalar function:

$$L(q'(t'), q', t') = L\left(g(q', t'), \frac{\partial g}{\partial q'}\left(\frac{\partial f}{\partial t'}\right)^{-1} \dot{q}' + \frac{\partial g}{\partial t'}\left(\frac{\partial f}{\partial t'}\right)^{-1}, f(t')\right)\left|\frac{\partial f}{\partial t'}\right| + \frac{dh(q'(t'), t')}{dt} \tag{11}$$

It turns out that this symmetry relation encodes valuable information about the properties of the system. In the case of continuous symmetries, this relation will enable us to discover non-trivial conserved quantities — i.e., combinations of the generalized coordinates whose values are conserved during the evolution of the system.

**Continuous Symmetry**

A continuous symmetry is a coordinate transformation that can be parameterized by a continuous parameter $\lambda$ such that the derivatives $\partial f / \partial \lambda$ and $\partial g / \partial \lambda$ exist and

$$\lim_{\lambda \to 0} f(t') = t', \quad \lim_{\lambda \to 0} g(q', t') = q'$$

**Conservation Laws**

For any continuous symmetry transformation, we can take the derivative of both sides of Eq. (11) with respect to $\lambda$ and the left hand side trivially vanishes. This yields

$$
\begin{aligned}
0 &= \frac{d}{d\lambda}\left[L\left(g(q',t'),\frac{\partial g}{\partial q'}\left(\frac{\partial f}{\partial t'}\right)^{-1}\dot{q}' + \frac{\partial g}{\partial t'}\left(\frac{\partial f}{\partial t'}\right)^{-1}, f(t')\right)\left|\frac{\partial f}{\partial t'}\right|\right]\Bigg|_{\lambda=0} \\
&= L\frac{\frac{\partial f}{\partial t'}}{\left|\frac{\partial f}{\partial t'}\right|}\left(\frac{\partial^2 f}{\partial t'\partial\lambda}\right) + \left\{\frac{\partial L}{\partial q_i}\frac{\partial g_i}{\partial\lambda} + \frac{\partial L}{\partial \dot{q}_i}\left[\left(\frac{\partial^2 g_i}{\partial q_j'\partial\lambda}\right)\left(\frac{\partial f}{\partial t'}\right)^{-1}\dot{q}_j - \frac{\partial g_i}{\partial q_j'}\left(\frac{\partial f}{\partial t'}\right)^{-2}\left(\frac{\partial^2 f}{\partial t'\partial\lambda}\right)\dot{q}_j\right.\right. \\
&\quad \left.\left.\left(\frac{\partial^2 g_i}{\partial t'\partial\lambda}\right)\left(\frac{\partial f}{\partial t'}\right)^{-1} - \frac{\partial g_i}{\partial t'}\left(\frac{\partial f}{\partial t'}\right)^{-2}\left(\frac{\partial^2 f}{\partial t'\partial\lambda}\right)\right] + \frac{\partial L}{\partial t}\frac{\partial f}{\partial\lambda}\right\}\left|\frac{\partial f}{\partial t'}\right|
\end{aligned}
$$

Our assumptions on the continuous nature of the symmetry yield some helpful simplifications. In particular, we see that by interchanging differentiation and limits as $\lambda \to 0$, the following identities appear:

$$
\frac{\partial f}{\partial t'}\Big|_{\lambda=0} = 1\,, \quad \frac{\partial g}{\partial q'}\Big|_{\lambda=0} = 1 \quad \frac{\partial g}{\partial t'}\Big|_{\lambda=0} = 0
$$

These identities yield a simplified version of the previous equation:

$$
0 = L\left(\frac{\partial^2 f}{\partial t'\partial\lambda}\right) + \frac{\partial L}{\partial q_i}\frac{\partial g_i}{\partial\lambda} + \frac{\partial L}{\partial\dot{q}_i}\left[\left(\frac{\partial^2 g_i}{\partial q_j'\partial\lambda}\right)\dot{q}_j - \left(\frac{\partial^2 f}{\partial t'\partial\lambda}\right)\dot{q}_j + \frac{\partial^2 g_i}{\partial t'\partial\lambda}\right] + \frac{\partial L}{\partial t}\frac{\partial f}{\partial\lambda}
$$

At this point, we use the Euler-Lagrange equations to replace $\partial L/\partial q_i$ with total time derivatives of $\partial L/\partial\dot{q}_i$. We then see that the entire right hand side can be written as the total time derivative of a scalar function:

$$
\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{q}_i}\frac{\partial g_i}{\partial\lambda} - \frac{\partial f}{\partial\lambda}\left(\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i - L\right)\right] = 0 \tag{12}
$$

which implies that dynamic variable inside the square brackets is a conserved quantity!

**Continuous Systems**

## 1.4 Appendix A: Change of variables theorem

Add it